Lecture 6. The notion of a function of one variable. Limit function. Limit theorems for functions. The concept of continuity of functions. The points of discontinuity.

A **real-valued function of a real variable** is a function that takes as input a real number, commonly represented by the variable *x*, for producing another real number, the *value* of the function, commonly denoted *f*(*x*). For simplicity, in this article a real-valued function of a real variable will be simply called a **function**. To avoid any ambiguity, the other types of functions that may occur will be explicitly specified.

Some functions are defined for all real values of the variables (one says that they are everywhere defined), but some other functions are defined only if the value of the variable is taken in a subset *X* of ℝ, the domain of the function, which is always supposed to contain an open subset of ℝ. In other words, a real-valued function of a real variable is a function

$$f : X \to \mathbb{R}$$

such that its domain *X* is a subset of ℝ that contains an open set.

A simple example of a function in one variable could be:

$$V : X \to \mathbb{R}$$
$$X = \{x \in \mathbb{R} \, : \, 0 \leq x\}$$
$$f(x) = \sqrt{x}$$

which is the square root of *x*.

Image

The **image** of a function $f(x)$ is the set of all values of *f* when the variable *x* runs in the whole domain of *f*. For a continuous (see below for a definition) real-valued function with a connected domain, the image is either an interval or a single value. In the latter case, the function is a constant function.

The preimage of a given real number *y* is the set of the solutions of the equation *y* = *f*(*x*).

Domain

The domain of a function of several real variables is a subset of ℝ that is sometimes, but not always, explicitly defined. In fact, if one restricts the domain *X* of a function *f* to a subset $Y \subset X$, one gets formally a different function, the *restriction* of *f* to *Y*, which is denoted $f_{|Y}$. In practice, it is often (but not always) not harmful to identify *f* and $f_{|Y}$, and to omit the subscript $_{|Y}$.

Conversely, it is sometimes possible to enlarge naturally the domain of a given function, for example by continuity or by analytic continuation. This means that it is not worthy to explicitly define the domain of a function of a real variable.

A **function** y = f (x) is a rule for determining y when we’re given a value of x. For example, the rule y = f (x) = 2x + 1 is a function. Any line y = mx + b is called a linear function. The graph of a function looks like a curve above (or below) the x-axis, where for any value of x the rule y = f (x) tells us how far to go above (or below) the x-axis to reach the curve.
Functions can be defined in various ways: by an algebraic formula or several algebraic formulas, by a graph, or by an experimentally determined table of values.

Given a value of x, a function must give at most one value of y. Thus, vertical lines are not functions. For example, the line x = 1 has infinitely many values of y if x = 1. It is also true that if x is any number not 1 there is no y which corresponds to x, but that is not a problem—only multiple y values is a problem.
In addition to lines, another familiar example of a function is the parabola $y = f(x) = x^2$. We can draw the graph of this function by taking various values of x (say, at regular intervals) and plotting the points $(x, f(x)) = (x, x^2)$. Then connect the points with a smooth curve. The two examples $y = f(x) = 2x + 1$ and $y = f(x) = x^2$ are both functions which can be evaluated at any value of x from negative infinity to positive infinity. For many functions, however, it only makes sense to take x in some interval or outside of some "forbidden" region. The interval of x-values at which we're allowed to evaluate the function is called the domain of the function.
For example, the square-root function $y = f(x) = \sqrt{x}$ is the rule which says, given an x-value, take the nonnegative number whose square is x. This rule only makes sense if x is positive or zero. We say that the domain of this function is $x \geq 0$, or more formally $\{x \in R \mid x \geq 0\}$. Alternately, we can use interval notation, and write that the domain is $[0, \infty)$. (In interval notation, square brackets mean that the endpoint is included, and a parenthesis means that the endpoint is not included.) The fact that the domain of $y = \sqrt{x}$ is $[0, \infty)$ means that in the graph of this function only above x-values on the right side of the x-axis.
Another example of a function whose domain is not the entire x-axis is: $y = f(x) = 1/x$, the reciprocal function. We cannot substitute x = 0 in this formula. The function makes sense, however, for any nonzero x, so we take the domain to be: $\{x \in R \mid x\ 6 = 0\}$. The graph of this function does not have any point (x, y) with x = 0. As x gets close to 0 from either side, the graph goes off toward infinity. We call the vertical line x = 0 an asymptote.
To summarize, two reasons why certain x-values are excluded from the domain of a function are that (i) we cannot divide by zero, and (ii) we cannot take the square root of a negative number. We will encounter some other ways in which functions might be undefined later.
Another reason why the domain of a function might be restricted is that in a given situation the x-values outside of some range might have no practical meaning. For example, if y is the area of a square of side x, then we can write $y = f(x) = x^2$. In a purely mathematical context the domain of the function $y = x^2$ is all of R. But in the story-problem context of finding areas of squares, we restrict the domain to positive values of x, because a square with negative or zero side makes no sense.
In a problem in pure mathematics, we usually take the domain to be all values of x at which the formulas can be evaluated. But in a story problem there might be further restrictions on the domain because only certain values of x are of interest or make practical sense.
In a story problem, often letters different from x and y are used. For example, the volume V of a sphere is a function of the radius r, given by the formula $V = f(r) = 4/3\pi r^3$
Also, letters different from f may be used. For example, if y is the velocity of something at time t, we may write y = v(t) with the letter v (instead of f) standing for the velocity function (and t playing the role of x).
The letter playing the role of x is called the independent variable, and the letter playing the role of y is called the dependent variable (because its value "depends on" the value of the independent variable).

Limit function.

The limit of a real-valued function of a real variable is as follows.[1] Let *a* be a point in topological closure of the domain *X* of the function *f*. The function, *f* has a limit *L* when *x* tends toward *a*, denoted

$$L = \lim_{x \to a} f(x),$$

if the following condition is satisfied: For every positive real number *ε* > 0, there is a positive real number *δ* > 0 such that

$$|f(x) - L| < \varepsilon$$

for all *x* in the domain such that

$$d(x, a) < \delta.$$

If the limit exists, it is unique. If *a* is in the interior of the domain, the limit exists if and only if the function is continuous at *a*. In this case, we have

$$f(a) = \lim_{x \to a} f(x).$$

When *a* is in the boundary of the domain of *f*, and if *f* has a limit at *a*, the latter formula allows to "extend by continuity" the domain of *f* to *a*.

Limit theorems for functions.

Theorem A. Suppose that **f** and **g** are functions such that **f(x) = g(x)** for all **x** in some open interval containing **a** except possibly for **a**, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

Theorem B. Suppose that **f** and **g** are functions such that the two limits

$$\lim_{x \to a} f(x) \text{ and } \lim_{x \to a} g(x)$$

exist, suppose that **k** is a constant and suppose that **n** is a positive integer. Then

1. $\lim_{x \to a} k = k$
2. $\lim_{x \to a} x = a$
3. $\lim_{x \to a} k \cdot f(x) = k \cdot \lim_{x \to a} f(x)$
4. $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
5. $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
6. $\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

7. $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}$ provided $\lim_{x\to a}g(x)\neq 0$.

8. $\lim_{x\to a}x^n=a^n$

9. $\lim_{x\to a}[f(x)]^n=\left[\lim_{x\to a}f(x)\right]^n$

10. $\lim_{x\to a}\sqrt[n]{x}=\sqrt[n]{a}$ provided $a>0$.

11. $\lim_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to a}f(x)}$ provided $\lim_{x\to a}f(x)>0$ when **n** is even.

12. If $f(x)\leq g(x)$ for all $x\neq a$ then $\lim_{x\to a}f(x)\leq\lim_{x\to a}g(x)$.

13. If **f** is a polynomial then $\lim_{x\to a}f(x)=f(a)$.

14. If **f** is a rational function then, for all **a** in the domain of **f**, $\lim_{x\to a}f(x)=f(a)$.

Theorem C. The limit

$$\lim_{x\to a}f(x)=M$$

if and only if the right-hand limits and left-hand limits exist and are equal to **M**:

$$\lim_{x\to a+}f(x)=\lim_{x\to a-}f(x)=M$$

Theorem D. (Squeeze Theorem) Suppose that **f**, **g** and **h** are three functions such that **f(x)** $\leq$ **g(x)** $\leq$ **h(x)** for all **x**. If

$$\lim_{x\to a}f(x)=\lim_{x\to a}h(x)=M$$

then

$$\lim_{x\to a}g(x)=M$$

Theorem E. Suppose that **f** and **g** are two functions such that

$\lim_{x\to a}f(x) = L \neq 0$ and $\lim_{x\to a}g(x)=0$

then the limit

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

does not exist.

The concept of continuity of functions.

For defining the continuity, it is useful to consider the distance function of ℝ, which is an everywhere defined function of 2 real variables: $d(x,y) = |x-y|$

A function *f* is **continuous** at a point a which is interior to its domain, if, for every positive real number *ε*, there is a positive real number *φ*such that $|f(x)-f(a)|<\epsilon$ for all x such that $d(x,a)<\varphi.$ In other words, *φ* may be chosen small enough for having the image by *f* of the interval of radius *φ* centered at a contained in the interval of length 2*ε* centered at $f(a).$ A function is continuous if it is continuous at every point of its domain.

The points of discontinuity.
A function that is **continuous**is a function whose graph has no breaks in it; i.e. it is a continuous curve. Generally speaking, **a function is continuous if you can draw its graph without picking up your pencil.** Notice, on the graph of y = sin(x), that the function is completely connected at all points. Many functions, however, will have isolated points where they are not connected. These problem points are called **discontinuities**.

Point discontinuities are also called **removable discontinuities** or**removable singularities**.

Sometimes we come across functions that are defined differently for a certain point. Consider the function $f(x) = \begin{cases} 1, & x = 3 \\ x^2, & all\ other\ real\ x - values \end{cases}$. We defined the value of the function to be 1 at the point x = 3, yet, the rest of the function is dictated by $f(x) = x^2$. We can see in the graph that the function is continuous except for the tiny hole in the curve at x = 1. It is discontinuous at a single point, and this discontinuity is called a **point discontinuity**.

In general, **point discontinuities occur when a function is defined specifically for an isolated x-value.** However, this does not guarantee a point discontinuity. For example, if we change our function slightly to $f(x) = \begin{cases} 9, & x = 3 \\ x^2, & all\ other\ real\ x - values \end{cases}$ it becomes continuous. This is because we have defined the value of the function at f(3) precisely to be the value of the function $f(x) = x^2$ at x = 3. In this case, we did not define the value at = 3 to be different from what it would be if the function were $f(x) = x^2$. Then there is no discontinuity. Compared to our last function with a point discontinuity, we moved the point back up to the function to "plug" up the hole, and it is now continuous. Always remember, if a function is defined like this, to check if the isolated point is a point discontinuity or just a trick.

Point Discontinuities also arise when our function has a denominator that can be equal to zero, but that part of the denominator can also be cancelled out with a like term in the numerator. Consider the function $f(x) = \frac{x^2(x-2)}{x-2}$. If we try to find the value of the function at x = 2, we end up getting $f(2) = \frac{2^2(2-2)}{2-2} = \frac{0}{0}$. 0/0 represents an undefined number - i.e. the function does not exist at that point. However, if we restrict the function to a domain that does not include x = 2, we can simply cancel out the $\frac{(x-2)}{(x-2)}$and be left with $f(x) = x^2$. This leaves us to define the function as $f(x) = \frac{x^2(x-2)}{(x-2)} = \begin{cases} x^2, & x \neq 2 \\ undefined, & x = 2 \end{cases}$. We have effectively removed the discontinuity to show that the function behaves exactly like $f(x) = x^2$, except at x = 2, where it is undefined.

In conclusion, **point discontinuities also occur when we can cancel a term in the denominator and the numerator**. They occur at the values for which the cancelled term is equal to zero. In our example, we removed x - 2, and x - 2 = 0 at x = 2. If we were to remove sin(x), we would have point discontinuities at integer multiples of π, since sin(π) = sin(2π) = sin(3π) = sin(nπ) = 0 for any integer n.